Exactness of the reduced crossed product functor for discrete groups (a very short survey)

A locally compact group G is called **exact** (or C*-exact) if for any short exact sequence of C*-algebras

$$0 \to I \to A \to A/I \to 0$$

where each algebra is endowed with a strongly continuous G-action making the sequence equivariant, the associated sequence of reduced crossed products

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is also exact. I.e. $-\rtimes_r G$ is an exact functor from the category of G-C*-algebras (with equivariant morphisms) to the category of C*-algebras. We induce a G-equivariant map $\phi : A \to B$ to the crossed product $\tilde{\phi} : A \rtimes_{(r)} G \to B \rtimes_{(r)} G$ by acting on a function $f \in C_c(G, A)$ by

$$\tilde{\phi}(f)(g) = \phi(f(g))$$

which clearly implies that $\tilde{\phi}(C_c(G, A)) \subset C_c(G, B)$ and after a little bit of work it can be shown to be a contractive *-preserving homomorphism with respect to both full and reduced C*-norm.

It is often possible to define exactness by looking at certain "maximally problematic" extensions or actions. For instance (see [9] Rem 2.11) a locally compact group G is exact if and only if it is exact on the equivariant extension

$$0 \to C_0(G) \to C_{cb}(G) \to C(\partial G) \to 0$$

where $C_{cb}(G)$ is the C*-algebra of bounded left uniformly continuous functions with the action of G by left translation.

Similarly, for a discrete group Γ, Kennedy and Kalantar have shown that Γ is exact if and only if the action of Γ is amenable on its Furstenberg boundary.

A third definition of exactness, which we will not mention in the sequel is the following: Let Γ be a countably generated discrete group, then Γ is exact if and only if it has Property A of Guoliang Yu.

Note that some authors call G exact if $C_r^*(G)$ is an exact C*-algebra, which means the minimal tensor product functor $-\otimes C_r^*(G)$ being exact. We will show later that these two definitions agree for discrete groups.

1 A word on crossed products

As opposed to the full crossed product functor, which is always exact, there are groups for which the reduced crossed product functor is not exact. However, one should note that for any locally compact group G the induced map

$$I \rtimes_r G \to A \rtimes_r G$$

is always injective, and

$$A \rtimes_r G \to (A/I) \rtimes_r G$$

is always surjective, so the sequence can only fail to be exact in the middle term. By the first property, to show that G is exact is equivalent to showing that

$$\frac{A \rtimes_r \Gamma}{I \rtimes_r \Gamma} \simeq (A/I) \rtimes_r \Gamma.$$

The left hand side also has a natural inclusion of $C_c(G, A/I)$ as a dense subalgebra, so to show exactness of the sequence amount to showing that the inclusion $C_c(G, A/I) \to \frac{A \rtimes_r \Gamma}{I \rtimes_r \Gamma}$ is norm-decreasing with respect to the reduced crossed product norm. It would then be an isomorphism by minimality of the reduced norm.

Adding to the confusion, though the full crossed product functor $- \rtimes G$ is an exact functor for any locally compact group G it does not automatically preserve injectivity for arbitrary G-equivariant *-homomorphisms. However, if $I \to A$ is an equivariant inclusion of an *ideal* of A, then any non-degenerate covariant representation $(\pi, U) : (I, G) \to B(H)$ extends to a non-degenerate covariant representation $(\tilde{\pi}, U)$ of (A, G), by the assignment

$$\pi(a)(\pi(b)v) = \pi(ab)(v) \qquad \text{for all } a \in A, b \in I \text{ and } v \in H.$$

This implies the induced map of the crossed products is injective. For more on this see [6].

For a reference, here are two pathological examples of non-exact groups

- Called by some authors the Gromov monsters, these are defined in [2] and are non-exact discrete groups. There seems to be no imbedding of these groups into some $B(H)$ for any Hilbert space H, so they are non-isomorphic to one will meet in practice.

- (Osajda) There are residually finite non-exact groups (i.e. groups G for which every $e \neq h \in G$ is contained in the complement of some normal finite index subgroup $N \subset G$). T

In [1] the following three useful results are proven -

- Any almost-connected locally compact group, (meaning a locally compact group G for which G/G^0 is compact) is exact.

- Any closed subgroups of exact groups are exact.

- If $N \subset G$ is a normal subgroup with both N and G/N exact, then G is exact.

The collection of exact groups is rather large and contains all the amenable groups. This is because if G is amenable, then any G-action is automatically an "amenable action" and we have $- \rtimes_r G = - \rtimes G$, the latter functor being exact. However, there are many non-amenable exact groups as the following theorem and subsequent comment shows:

Theorem 1 ((Guentner, Higson, Weinberger) [4])**.** *Let K be a field and let n be a positive integer and $G \subset GL_n(K)$ any subgroup. Then the reduced group C*-algebra $C_r^*(G)$ is exact.*

Now it turns out that if Γ is any *discrete* group we have

$$C_r^*(\Gamma) \text{ is an exact C*-algebra} \quad \Leftrightarrow \quad - \rtimes_{r,\alpha} \Gamma \text{ is an exact functor}$$

The $\Leftarrow$ implication is always true for any locally compact group since if Γ acts trivial on a C*-algebra A, then $A \otimes_{min} C_r^*(\Gamma) \simeq A \rtimes_r \Gamma$. The statement is proved in [1][Theorem 5.2]. We can summarize the above, by the following Theorem

Theorem 2. *If K is any field and Γ is a discrete linear subgroup of $GL_n(K)$, the functor $- \rtimes_r \Gamma$ is exact.*

Hence most discrete groups one will meet in the wild are exact. The proof of Theorem 5.2 however says something slightly stronger. It states that for a discrete group Γ the sequence

$$0 \to I \rtimes_r \Gamma \to A \rtimes_r \Gamma \to (A/I) \rtimes_r \Gamma \to 0 \tag{1}$$

is exact if and only if the sequence

$$0 \to (I \rtimes \Gamma) \otimes C_r^\star(\Gamma) \to (A \rtimes \Gamma) \otimes C_r^\star(\Gamma) \to [(A/I) \rtimes \Gamma] \otimes C_r^\star(\Gamma) \to 0 \tag{2}$$

is exact, where the tensor product is the minimal one. From this one can show that the sequence (1) is exact in many cases where the group Γ is not exact. Let us list some of them -

Corollary 1. *For Γ any discrete group, the sequence (1) is exact in the following cases*

1. *The action of Γ is amenable on all C*-algebras in the sequence.*

2. *The C*-algebras $I \rtimes \Gamma$, $A \rtimes \Gamma$ and $(A/I) \rtimes \Gamma$ are all nuclear.*

3. *There is a unique C*-norm on the algebraic tensor product $(A/I) \rtimes \Gamma \odot C_r^\star(\Gamma)$.*

4. *A/I is nuclear and the action of Γ on A/I is amenable.*

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5. *The sequence* $0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0$ *is locally split (or semisplit).*

6. *The sequence* $0 \to I \to A \to A/I \to 0$ *is equivariantly locally split (or equivariantly semisplit).*

Case 1) follows again from the fact that for any Γ-C*-algebra B with an amenable Γ action we have $B \rtimes_r \Gamma = B \rtimes \Gamma$, and $- \rtimes \Gamma$ is an exact functor.

Case 2 follows from the correspondence of the sequences (1) and (2), together with the fact that the minimal tensor product with a fixed C*-algebra sends exact sequences of *nuclear* C*-algebras to exact sequences of C*-algebras (being equivalent to the maximal tensor product, which is exact).

Case 3) follows follows from the correspondence of the sequences (1) and (2) and Corollary 3.7.3 [5], which states that the sequence

$$0 \to I \otimes B \to A \otimes B \to (A/I) \otimes B \to 0$$

is exact if the algebraic tensor product $(A/I) \odot B$ admits a unique C*-norm, which happens for instance when either (A/I) or B are nuclear.

Case 4): The conditions assures that the crossed product $(A/I) \rtimes \Gamma$ is nuclear, hence the claim follows from Case 3 above.

Note that $B \rtimes \Gamma$ is nuclear always implies that B is nuclear. If the action is amenable, the converse also holds. For transformation groupoids given by the action of a discrete group on a locally compact Hausdorff space, a theorem of Anantharaman-Delaroche states that $A \rtimes_r \Gamma$ is nuclear if and only if the action of Γ is amenable.

Case 5) If the sequence $0 \to I \to A \to A/I \to 0$ is locally split (or semisplit), then by Proposition 3.7.6 of [5] the sequence

$$0 \to I \otimes B \to A \otimes B \to (A/I) \otimes B \to 0$$

exact for any C*-algebra B.

Case 6) If the sequence $0 \to I \to A \to A/I \to 0$ is equivariantly locally split/semisplit, then the sequence $0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0$ is locally split/semisplit respectively, so we can use Case 4).

For the proof of these facts and many more see [5].

A word of caution: Contrary to what seems to be the main theorem of [8], there are exact sequences which are semisplit but not equivariantly semisplit. Even if one restricts to finitely generated groups and extensions of commutative C*-algebras. This follows from the example of section 7 of [7] where an extension of commutative Γ C*-algebras yields a non-exact extension of crossed products, which (using Case 4 above), implies the original extension cannot be equivariantly semisplit.

References

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